HERMITE POLYNOMIALS

HERMITE'S EQUATION AND ITS SOLUTION

Hermite's equation is given by

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0, \tag{5.1}$$

Hermite polynomial of order n:

$$H_n(x) = \sum_{r=0}^{[\frac{1}{2}n]} (-1)^r \frac{n!}{r!(n-2r)!}(2x)^{n-2r}.$$

GENERATING FUNCTION

Theorem 5.1

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x).$$

PROOF

We wish to pick out the coefficient of t^n in the power series expansion of $\exp(2tx - t^2)$.

Now,

$$\begin{aligned} e^{2tx-t^2} &= e^{2tx}\, e^{-t^2} \\ &= \sum_{r=0}^{\infty} \frac{(2tx)^r}{r!} \sum_{s=0}^{\infty} \frac{(-t^2)^s}{s!} \\ &= \sum_{r,s=0}^{\infty} (-1)^s \frac{(2x)^r}{r!s!} t^{r+2s}. \end{aligned}$$

For a fixed value of s we obtain t^n by taking $r + 2s = n$, i.e., $r = n - 2s$, so that for this value of s the coefficient of t^n is just given by

$$(-1)^s \frac{(2x)^{n-2s}}{(n-2s)!s!}.$$

The total coefficient of t^n is obtained by summing over all allowed values of s, and, since $r = n - 2s$, this implies that we must have $n - 2s \geqslant 0$, i.e., $s \leqslant \frac{1}{2}n$. Thus, if n is even, s goes from 0 to $\frac{1}{2}n$, while if n is odd, s goes from 0 to $\frac{1}{2}(n-1)$; that is, in all cases, s goes from 0 to $[\frac{1}{2}n]$ with $[\frac{1}{2}n]$ defined as above.

Thus we have:

$$\begin{aligned}\text{coefficient of } t^n &= \sum_{s=0}^{[\frac{1}{2}n]} (-1)^s \frac{1}{(n-2s)!s!}(2x)^{n-2s} \\ &= \frac{1}{n!}H_n(x),\end{aligned}$$

(by definition (5.3)).

Theorem 5.2

$$H_n(x) = (-1)^n \, e^{x^2} \frac{d^n}{dx^n} \, e^{-x^2}.$$

PROOF

By use of the generating function of theorem 5.1 and Taylor's theorem, which states that

$$F(t) = \sum_{n=0}^{\infty} \left(\frac{d^n F}{dt^n}\right)_{t=0} \frac{t^n}{n!},$$

we have

$$\begin{aligned}H_n(x) &= \left[\frac{\partial^n}{\partial t^n} \, e^{2tx-t^2}\right]_{t=0} \\ &= \left[\frac{\partial^n}{\partial t^n} \, e^{x^2-(x-t)^2}\right]_{t=0} \\ &= e^{x^2}\left[\frac{\partial^n}{\partial t^n} \, e^{-(x-t)^2}\right]_{t=0}.\end{aligned}$$

But

$$\frac{\partial}{\partial t}f(x-t) = -\frac{\partial}{\partial x}f(x-t),$$

so that

$$\frac{\partial^n}{\partial t^n}f(x-t) = (-1)^n \frac{\partial^n}{\partial x^n}f(x-t)$$

and we have

$$H_n(x) = (-1)^n e^{x^2}\left[\frac{\partial^n}{\partial x^n} e^{-(x-t)^2}\right]_{t=0}$$

$$= (-1)^n e^{x^2}\frac{\partial^n}{\partial x^n} e^{-x^2}$$

$$= (-1)^n e^{x^2}\frac{d^n}{dx^n} e^{-x^2}$$

For the first few orders we obtain

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2x \\ H_2(x) &= 4x^2 - 2 \\ H_3(x) &= 8x^3 - 12x \\ H_4(x) &= 16x^4 - 48x^2 + 12 \\ H_5(x) &= 32x^5 - 160x^3 + 120x. \end{aligned}$$

ORTHOGONALITY PROPERTIES OF THE HERMITE POLYNOMIALS

Theorem 5.5

$$\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)\, dx = 2^n n!(\sqrt{\pi})\delta_{nm}.$$

PROOF

We have

$$e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$$

and

$$e^{-s^2+2sx} = \sum_{m=0}^{\infty} H_m(x)\frac{s^m}{m!}$$

so that $\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x)\, dx$ is the coefficient of $(t^n s^m)/(n!m!)$ in the expansion of $\int_{-\infty}^{\infty} e^{-x^2}\, e^{-t^2+2tx}\, e^{-s^2+2sx}\, dx$.

But

$$\int_{-\infty}^{\infty} e^{-x^2}\, e^{-t^2+2tx}\, e^{-s^2+2sx}\, dx$$

$$= e^{-t^2-s^2} \int_{-\infty}^{\infty} \exp\{-x^2 + 2(s+t)x\}\, dx$$

$$= e^{-t^2-s^2} \int_{-\infty}^{\infty} \exp[-\{x-(s+t)\}^2 + (s+t)^2]\, dx$$

$$= e^{2st} \int_{-\infty}^{\infty} \exp[-\{x-(s+t)\}^2]\, dx$$

$$= e^{2st} \int_{-\infty}^{\infty} \exp(-u^2)\, du$$

(changing the variable of integration to $u = x - (s+t)$)

$$= e^{2st} \sqrt{\pi}$$

(by the corollary to theorem 2.6)

$$= \sum_{n=0}^{\infty} (\sqrt{\pi}) \frac{2^n s^n t^n}{n!}.$$

Hence the coefficient of $(t^n s^m)/(n!m!)$ is zero if $m \neq n$ and is $(\sqrt{\pi})2^n n!$ when $m = n$.

Hence $$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ (\sqrt{\pi})2^n n! & \text{if } m = n \end{cases}$$

or, making use of the Kronecker delta,

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = (\sqrt{\pi})2^n n!\,\delta_{mn}.$$

Example 3

Show that $P_n(x) = \dfrac{2}{(\sqrt{\pi})n!}\displaystyle\int_0^{\infty} t^n\, e^{-t^2} H_n(xt)\, dt.$

From equation (5.3) we have

$$H_n(xt) = \sum_{r=0}^{[\frac{1}{2}n]} (-1)^r \frac{n!}{r!(n-2r)!}(2xt)^{n-2r}$$

so that

$$\frac{2}{(\sqrt{\pi})n!}\int_0^{\infty} t^n\, e^{-t^2} H_n(xt)\, dt$$

$$= \frac{2}{(\sqrt{\pi})n!}\int_0^{\infty} t^n\, e^{-t^2} \sum_{r=0}^{[\frac{1}{2}n]} (-1)^r \frac{n!}{r!(n-2r)!} 2^{n-2r} x^{n-2r} t^{n-2r}\, dt$$

$$= \sum_{r=0}^{[\frac{1}{2}n]} \frac{2^{n-2r+1}(-1)^r x^{n-2r}}{(\sqrt{\pi})r!(n-2r)!}\int_0^{\infty} e^{-t^2}\, t^{2n-2r}\, dt$$

$$= \sum_{r=0}^{[\frac{1}{2}n]} \frac{2^{n-2r+1}(-1)^r x^{n-2r}}{(\sqrt{\pi})r!(n-2r)!}\, \tfrac{1}{2}\Gamma(n-r+\tfrac{1}{2})$$

(by theorem 2.4)

$$= \sum_{r=0}^{[\frac{1}{2}n]} \frac{2^{n-2r}(-1)^r x^{n-2r}}{(\sqrt{\pi})r!(n-2r)!}\,\frac{(2n-2r)!}{2^{2n-2r}(n-r)!}\sqrt{\pi}$$

(by the corollary to theorem 2.10)

$$= \sum_{r=0}^{[\frac{1}{2}n]} (-1)^r \frac{(2n-2r)!}{2^n r!(n-2r)!(n-r)!} x^{n-2r}$$

$= P_n(x),$

(by equation (3.17)).

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