BESSEL FUNCTIONS

Bessel's equation of order n is

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \tag{4.1}$$

(where, since it is only n^2 that enters the equation, we may always take n to be non-negative).

we obtain the solution which we shall denote by $J_n(x)$ and shall call the

Bessel function of the first kind of order n:

$$J_n(x)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}.$$

So far we have dealt with the root of the indicial equation $s = n$. The other root $s = -n$ will also give a solution of Bessel's equation, and the form of this solution is obtained just by replacing n in all the equations above by $-n$, so that we obtain the solution to Bessel's equation

$$J_{-n}(x)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{2r-n}. \tag{4.7}$$

Theorem 4.1

When n is an integer (positive or negative),

$$J_{-n}(x)=(-1)^nJ_n(x).$$

Proof

First consider $n > 0$.
Then

$$J_{-n}(x)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{2r-n}$$

from equation (4.7).

But $\Gamma(-n+r+1)$ is infinite (and hence $1/\{\Gamma(-n+r+1)\}$ is zero) for those values of r which make the argument a negative integer or zero, i.e., for $r = 0, 1, 2, \ldots (n-1)$ (remembering that this is possible because n is integral).

Hence the sum over r in the above expression for $J_{-n}(x)$ can equally well be taken from n to infinity, and then

$$J_{-n}(x) = \sum_{r=n}^{\infty} (-1)^r \frac{1}{r!\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n}$$

$$= \sum_{m=0}^{\infty} (-1)^{m+n} \frac{1}{(m+n)!\Gamma(m+1)} \left(\frac{x}{2}\right)^{2(m+n)-n}$$

(where we have changed the variable of summation to $m = r - n$)

$$= (-1)^n \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m+n)!\Gamma(m+1)} \left(\frac{x}{2}\right)^{2m+n}.$$

But

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m+n}$$

(by equation (4.6))

so all that remains in order to complete the proof is to show that

$$(m+n)!\Gamma(m+1) = m!\Gamma(n+m+1)$$

for n and m integral.

But

$$(m+n)!\Gamma(m+1) = (m+n)(m+n-1)\ldots(m+1)m!\Gamma(m+1)$$
$$= m!\Gamma(m+n+1)$$

(on using repeatedly the result that $\Gamma(x+1) = x\Gamma(x)$), and thus the result is proved.

Now consider $n < 0$; in this case we may write $n = -p$ with $p > 0$. Then what we require to prove is that

$$J_p(x) = (-1)^{-p} J_{-p}(x)$$

or

$$(-1)^p J_p(x) = J_{-p}(x)$$

which, of course, since p is positive, is just the result we have proved above.

Let us summarise what we have proved so far: we have shown that if n is not an integer then $J_n(x)$ and $J_{-n}(x)$ (defined by equations (4.6) and (4.7), respectively) are independent solutions of Bessel's equation (so that the general solution is given by $AJ_n(x) + BJ_{-n}(x)$) while if n is an integer they are still solutions of Bessel's equation but are related by

$$J_{-n}(x) = (-1)^n J_n(x).$$

GENERATING FUNCTION FOR THE BESSEL FUNCTIONS

Theorem 4.5

$$\exp\left\{\frac{1}{2}x\left(t-\frac{1}{t}\right)\right\} = \sum_{n=-\infty}^{\infty} t^n J_n(x).$$

PROOF

We expand $\exp\left\{\frac{1}{2}x\left(t-\frac{1}{t}\right)\right\}$ in powers of t and show that the coefficient of t^n is $J_n(x)$:

$$\exp\left\{\frac{1}{2}x\left(t-\frac{1}{t}\right)\right\} = \exp\left(\frac{1}{2}xt\right).\exp\left(-\frac{1}{2}\frac{x}{t}\right)$$

$$= \sum_{r=0}^{\infty}\frac{(\frac{1}{2}xt)^r}{r!}\sum_{s=0}^{\infty}\frac{(-\frac{1}{2}x/t)^s}{s!}$$

$$= \sum_{r,\,s=0}^{\infty}\frac{(\frac{1}{2})^r x^r t^r(-1)^s(\frac{1}{2})^s x^s t^{-s}}{r!\,s!}$$

$$= \sum_{r,\,s=0}^{\infty}(-1)^s\left(\frac{1}{2}\right)^{r+s}\frac{x^{r+s}t^{r-s}}{r!\,s!}. \qquad (4.12)$$

We now pick out the coefficient of t^n, where first we consider $n \geqslant 0$. For a fixed value of r, to obtain the power of t as t^n we must have $s = r - n$. Thus for this particular value of r the coefficient of t^n is

$$(-1)^{r-n}\left(\frac{1}{2}\right)^{2r-n}\frac{x^{2r-n}}{r!(r-n)!}.$$

We get the total coefficient of t^n by summing over all allowed values of r. Since $s = r - n$ and we require $s \geqslant 0$, we must have $r \geqslant n$. Hence the total coefficient of t^n is

$$\sum_{r=n}^{\infty}(-1)^{r-n}\left(\frac{1}{2}\right)^{2r-n}\frac{x^{2r-n}}{r!(r-n)!} = \sum_{p=0}^{\infty}(-1)^p\frac{(x/2)^{2p+n}}{(p+n)!\,p!}$$

(where we have set $p = r - n$)

$$= \sum_{p=0}^{\infty}(-1)^p\frac{(x/2)^{2p+n}}{\Gamma(p+n+1)p!}$$

(remembering that both p and n are integral, so that we may use the result of theorem 2.3 that $\Gamma(p+n+1) = (p+n)!$)

$$= J_n(x)$$

(by equation (4.6)).

If now $n < 0$, we still have the coefficient of t^n for a fixed value of r given by

$$(-1)^{r-n}\left(\frac{1}{2}\right)^{2r-n}\frac{x^{2r-n}}{r!(r-n)!}$$

but now the requirement that $s \geqslant 0$ with $s = r - n$ is satisfied for *all* values of r. Thus the coefficient of t^n is just

$$\begin{aligned}\sum_{r=0}^{\infty}(-1)^{r-n}\left(\frac{1}{2}\right)^{2r-n}\frac{x^{2r-n}}{r!(r-n)!} &= (-1)^{-n}\sum_{r=0}^{\infty}(-1)^r\frac{(x/2)^{2r-n}}{r!\Gamma(r-n+1)}\\ &= (-1)^n J_{-n}(x)\\ &\qquad \text{(by equation (4.7))}\\ &= J_n(x)\\ &\qquad \text{(by theorem 4.1).}\end{aligned}$$

INTEGRAL REPRESENTATIONS FOR BESSEL FUNCTIONS

Theorem 4.6

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos(n\phi - x\sin\phi)\,\mathrm{d}\phi$$

(n integral).

PROOF

Since $J_{-n}(x) = (-1)^n J_n(x)$ for n integral, the result of theorem 4.5 may be written in the form

$$\exp\left\{\frac{1}{2}x\left(t-\frac{1}{t}\right)\right\} = J_0(x) + \sum_{n=1}^{\infty}\{t^n + (-1)^n t^{-n}\}J_n(x).$$

If we now write $t = \mathrm{e}^{\mathrm{i}\phi}$ so that

$$t - \frac{1}{t} = \mathrm{e}^{\mathrm{i}\phi} - \mathrm{e}^{-\mathrm{i}\phi} = 2\mathrm{i}\sin\phi$$

this equation becomes

$$\mathrm{e}^{\mathrm{i}x\sin\phi} = J_0(x) + \sum_{n=1}^{\infty}\{\mathrm{e}^{\mathrm{i}n\phi} + (-1)^n\mathrm{e}^{-\mathrm{i}n\phi}\}J_n(x).$$

But when n is even

$$\mathrm{e}^{\mathrm{i}n\phi} + (-1)^n\,\mathrm{e}^{-\mathrm{i}n\phi} = \mathrm{e}^{\mathrm{i}n\phi} + \mathrm{e}^{-\mathrm{i}n\phi} = 2\cos n\phi,$$

while when n is odd we have

$$e^{in\phi} + (-1)^n e^{-in\phi} = e^{in\phi} - e^{-in\phi} = 2i \sin n\phi.$$

Thus we have

$$\begin{aligned} e^{ix\sin\phi} &= J_0(x) + \sum_{n \text{ even}} 2\cos n\phi\, J_n(x) + \sum_{n \text{ odd}} 2i \sin n\phi\, J_n(x) \\ &= J_0(x) + \sum_{k=1}^{\infty} 2\cos 2k\phi\, J_{2k}(x) + i \sum_{k=1}^{\infty} 2\sin(2k-1)\phi\, J_{2k-1}(x). \end{aligned}$$

Equating real and imaginary parts of this equation gives

$$\cos(x\sin\phi) = J_0(x) + \sum_{k=1}^{\infty} 2\cos 2k\phi\, J_{2k}(x) \tag{4.13}$$

$$\sin(x\sin\phi) = \sum_{k=1}^{\infty} 2\sin(2k-1)\phi\, J_{2k-1}(x). \tag{4.14}$$

If we multiply both sides of equation (4.13) by $\cos n\phi$ ($n \geqslant 0$), both sides

of equation (4.14) by $\sin n\phi$ ($n \geqslant 1$), integrate from 0 to π and use the identities

$$\int_0^{\pi} \cos m\phi \cos n\phi \, d\phi = \begin{cases} 0 & (m \neq n) \\ \pi/2 & (m = n \neq 0) \\ \pi & (m = n = 0) \end{cases}$$

and

$$\int_0^{\pi} \sin m\phi \sin n\phi \, d\phi = \begin{cases} 0 & (m \neq n) \\ \pi/2 & (m = n \neq 0), \end{cases}$$

we obtain the results

$$\int_0^{\pi} \cos n\phi \cos(x\sin\phi)\, d\phi = \begin{cases} \pi J_n(x) & (n \text{ even}) \\ 0 & (n \text{ odd}) \end{cases}$$

and

$$\int_0^{\pi} \sin n\phi \sin(x\sin\phi)\, d\phi = \begin{cases} 0 & (n \text{ even}) \\ \pi J_n(x) & (n \text{ odd}). \end{cases}$$

Adding these last two equations gives

$$\int_0^{\pi} \{\cos n\phi \cos(x\sin\phi) + \sin n\phi \sin(x\sin\phi)\}\, d\phi = \pi J_n(x)$$

for all positive integral n.

Hence
$$\int_0^\pi \cos(n\phi - x\sin\phi)\,\mathrm{d}\phi = \pi J_n(x)$$

which is the required result for positive n.

If n is negative, we may set $n = -m$ where m is positive, so that the required result is

$$\int_0^\pi \cos(-m\phi - x\sin\phi)\,\mathrm{d}\phi = \pi J_{-m}(x)$$

(where m is positive).

But
$$\int_0^\pi \cos(-m\phi - x\sin\phi)\,\mathrm{d}\phi$$

$$= \int_\pi^0 \cos\{-m(\pi - \theta) - x\sin(\pi - \theta)\}.\ -\mathrm{d}\theta$$

(where we have changed the variable by setting $\theta = \pi - \phi$)

$$= \int_0^\pi \cos\{-m\pi + m\theta - x\sin\theta\}\,\mathrm{d}\theta$$

$$= \int_0^\pi \{\cos(m\theta - x\sin\theta)\cos m\pi + \sin(m\theta - x\sin\theta)\sin m\pi\}\,\mathrm{d}\theta$$

$$= (-1)^m \int_0^\pi \cos(m\theta - x\sin\theta)\,\mathrm{d}\theta$$

$$= (-1)^m \pi J_m(x)$$

(since we know the result to be true for positive m)

$$= \pi J_{-m}(x)$$

$$= \pi J_n(x).$$

Theorem 4.27

$$\int_0^\infty \mathrm{e}^{-ax} J_0(bx)\,\mathrm{d}x = \frac{1}{\sqrt{(a^2 + b^2)}} \qquad (a > 0).$$

Proof

From theorem 4.6 we have

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\phi)\,\mathrm{d}\phi.$$

Hence

$$\int_0^\infty e^{-ax}J_0(bx)\,\mathrm{d}x = \int_0^\infty e^{-ax}\frac{1}{\pi}\int_0^\pi \cos(bx\sin\phi)\,\mathrm{d}\phi\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^\pi\left[\int_0^\infty e^{-ax}\,\tfrac{1}{2}\{\exp(ibx\sin\phi) + \exp(-ibx\sin\phi)\}\,\mathrm{d}x\right]\mathrm{d}\phi$$

(interchanging the order of integration)

$$= \frac{1}{2\pi}\int_0^\pi\left[\frac{\exp\{-(a - ib\sin\phi)x\}}{-a + ib\sin\phi} + \frac{\exp\{-(a + ib\sin\phi)x\}}{-a - ib\sin\phi}\right]_0^\infty \mathrm{d}\phi$$

$$= \frac{1}{2\pi}\int_0^\pi\left\{\frac{1}{a - ib\sin\phi} + \frac{1}{a + ib\sin\phi}\right\}\mathrm{d}\phi$$

$$= \frac{a}{\pi}\int_0^\pi \frac{1}{a^2 + b^2\sin^2\phi}\,\mathrm{d}\phi.$$

But this last integral may be evaluated by elementary means (e.g., by the substitution $u = \cot\phi$) to give

$$\int_0^\infty e^{-ax}J_0(bx)\,\mathrm{d}x = \frac{a}{\pi}\,\frac{\pi}{a\sqrt{(a^2 + b^2)}} = \frac{1}{\sqrt{(a^2 + b^2)}}.$$

Theorem 4.28

$$\int_0^\infty J_n(bx)\,\mathrm{d}x = \frac{1}{b} \quad \textit{(if } n \textit{ is a non-negative integer).}$$

Proof

We first prove the result for $n = 0$ and $n = 1$, and then show that if the result is true for $n = N$, it is also true for $n = N + 2$, thus proving the result for all non-negative integral n.

For $n = 0$ we take the limit as $a \to 0$ of the result of theorem 4.27, obtaining

$$\int_0^\infty J_0(bx)\,\mathrm{d}x = \frac{1}{b}.$$

For $n = 1$ we make use of theorem 4.8(ii), which says t

$$\frac{d}{dx}\{x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x)$$

so that, by taking $n = 0$, we have

$$\frac{d}{dx}J_0(x) = -J_1(x),$$

and replacing x by bx gives

$$\frac{d}{d(bx)}J_0(bx) = -J_1(bx)$$

which is equivalent to

$$\frac{1}{b}\frac{d}{dx}J_0(bx) = -J_1(bx).$$

Hence $$\int_0^\infty J_1(bx)\,dx = -\frac{1}{b}\Big[J_0(bx)\Big]_0^\infty$$

$$= \frac{1}{b},$$

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